Fractional Linear Transformations

Problem: Find a fractional linear transformation which send (z_0, z_1, z_2) to (w_0, w_1, w_2)

Solution: Let $\zeta = \dfrac{z - z_0}{z - z_2}\;\dfrac{z_1 - z_2}{z_1 - z_0}$ $\quad (z_0, z_1, z_2) \longmapsto (0, 1, \infty)$

$$\zeta = \frac{w - w_0}{w - w_2}\;\frac{w_1 - w_2}{w_1 - w_0} \qquad (w_0, w_1, w_2) \to (0, 1, \infty)$$

Solve second equation for w in terms of ζ and then replace ζ by right side of first equation.

Shortcut: Set right sides equal, and solve for w in terms of z.

Example (See II.7 e))

$(1+i, 2, 0) \longmapsto (0, \infty, i-1)$

$$\frac{w - 0}{w - (i-1)} \cdot \cancel{\frac{\infty - (i-1)}{\infty - 0}} = \frac{z - (1+i)}{z - 0}\;\frac{(2-0)}{2-(1+i)}$$

$$\frac{w}{w - (i-1)} = \frac{z - (1+i)}{z} \cdot \frac{2}{1-i}$$

$$wz(1-i) = (w + (1-i))(z - (1+i))2$$

$$= 2wz - 2w(1+i) + 2(1-i)z - 4$$

$$w(z(1-i) - 2z + 2(1+i)) = 2(1-i)z - 4$$

$$w = \frac{2(1-i)z - 4}{-(1+i)z + 2(1+i)} \qquad \boxed{\text{You can stop here!}}$$

Simplification: divide numerator + denominator by $1+i$

$$= \frac{2\left(\frac{1-i}{1+i}\right)z - \frac{4}{(1+i)}}{-z+2} = \frac{2(-i)z - 2(1-i)}{-z+2}$$

$$= \frac{(2i)z + 2(1-i)}{z-2} = \frac{2i\left(z + \frac{1-i}{i}\right)}{z-2} = 2i\,\frac{(z-i-1)}{z-2}$$